Math 429 - Exercise Sheet 3

Recall that a group G is called **abelian** if $gh = hg$ for all $g, h \in G$, or in other words, if $G = Z(G)$.

1. (a) Show that if a Lie group G is abelian, then its Lie algebra $\mathfrak{g}$ is also abelian.

(b) Show that the converse also holds if G is connected.

2. Show that if a Lie group G is abelian, then the exponential map

$$\exp : \mathfrak{g} \to G$$

is surjective. Conclude that $G = \mathfrak{g}/\Gamma$ for a discrete additive subgroup $\Gamma \subset \mathfrak{g}$.

*(above, **discrete** refers to the topological property, i.e. in which any point is its own open subset; we say that X is a discrete subset of a topological space M if the latter can be covered by open subsets which intersect X in either zero or one point).*

3. The standard example of the situation in the previous problem is $\mathfrak{g} = \mathbb{R}$ and $G = S_1$. What about in the complex case, what are the possible complex abelian Lie groups G with $\mathfrak{g} = \mathbb{C}$?

Recall that a topological space M is called **simply connected** if it is

- (path)-connected, i.e. any two points of M can be joined by a (continuous) path $\gamma : [0, 1] \to M$
- any two paths $\gamma, \gamma' : [0, 1] \to M$ with the same start and end points $\gamma(0) = \gamma'(0), \gamma(1) = \gamma'(1)$ can be related by a (continuous) **homotopy**

  $$H : [0, 1] \times [0, 1] \to M$$

  such that $H(t, 0) = \gamma(t)$, $H(t, 1) = \gamma'(t)$ and $H(0, t) = \gamma(0) = \gamma'(0)$, $H(1, t) = \gamma(1) = \gamma'(1)$ for all t (intutively, we think of $H_t = H(-, t)$ as giving us a family of paths that connect the paths γ and γ', all the while keeping the endpoints fixed).

4. Work out the details in Theorem 4.(b) in class.